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A map colour theorem for the union of graphs

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Abstract

In 1890 Heawood [Map colour theorem, Quart. J. Pure Appl. Math. 24 (1890) 332–338] established an upper bound for the chromatic number of a graph embedded on a surface of Euler genus $g \geqslant 1$. This upper bound became known as the Heawood number $H(g)$. Almost a century later, Ringel [Map Color Theorem, Springer, New York, 1974] and Ringel and Youngs [Solution of the Heawood map-coloring problem, Proc. Nat. Acad. Sci. USA 60 (1968) 438–445] proved that the Heawood number $H(g)$ is in fact the maximum chromatic number as well as the maximum clique number of graphs embedded on a surface of Euler genus $g \geqslant 1$ besides the Klein bottle. In this paper, we present a Heawood-type formula for the edge disjoint union of two graphs that are embedded on a given surface Σ. More precisely, we determine the number $H_2(\Sigma)$ such that if a graph G embedded on Σ is the edge disjoint union of two graphs G_1 and G_2, then

$$\omega(G_1) + \omega(G_2) \leqslant \chi(G_1) + \chi(G_2) \leqslant H_2(\Sigma).$$

Similar to the results of Ringel and Ringel and Youngs, we show that this bound is sharp for all but at most one non-orientable surface Σ.

Keywords: Graph decomposition; Chromatic number; Surface embedding

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1. Introduction

1.1. Motivation and main results

All the graphs considered are finite and simple. Let G be a graph and let $k \geqslant 1$ be an integer. A *k-decomposition* $(G_1, \dots, G_k)$ of G is a partition of its edge set to form k spanning subgraphs $G_1, \dots, G_k$. That is, each G_i has the same vertex set as G, and every edge of G belongs to exactly one of $G_1, \dots, G_k$. Such decompositions can be interpreted as unrestricted k-edge-colourings of G. Moreover, (G_1, G_2) is a 2-decomposition of the complete graph K_n if and only if the graph G_1 is the complement of the graph G_2 and has n vertices.

For a graph parameter p, a positive integer k, and a graph G, let

$$p(k; G) = \max\left\{\sum_{i=1}^{k} p(G_i) \mid (G_1, \dots, G_k) \text{ is a } k\text{-decomposition of } G\right\}.$$

The parameters that interest us are the clique number ω, the chromatic number χ, the list-chromatic number χ^ℓ, and the colouring number σ, where $\sigma(G) = 1 + \max_{H \subseteq G} \delta(H)$.

Nordhaus and Gaddum [14] proved that $\chi(2; K_n) = n + 1$. Moreover, it is known that $\omega(2; K_n) = \chi(2; K_n) = \chi^\ell(2; K_n) = \sigma(2; K_n) = n + 1$. For a short proof, see [9]. This paper deals mainly with decompositions of the complete graph K_n and establishes lower and upper bounds for $p(k; K_n)$ where $p \in \{\omega, \chi, \chi^\ell, \sigma\}$.

The present paper is concerned with graphs embedded on a given (closed) surface Σ. For a graph parameter p and an integer $k \geqslant 1$, let

$$p(k; \Sigma) = \max\{p(k; G) \mid G \text{ is embedded on } \Sigma\}.$$

Surfaces can be classified according to their genus and orientability. For $h \geqslant 0$, the *orientable surfaces* Σ_h are obtained by adding h handles to a sphere. For $h \geqslant 1$, the *non-orientable surfaces* Π_h are obtained from a sphere with h holes by attaching h Möbius bands along their boundaries to the boundaries of the holes. For example, Π_1 is the projective plane, Π_2 is the Klein bottle, etc. The *Euler genus* $g(\Sigma)$ of the surface Σ is $2h$ if $\Sigma = \Sigma_h$ and is h if $\Sigma = \Pi_h$. Then $2 - g(\Sigma)$ is the *Euler characteristic* of Σ.

Consider a simple graph G with vertex set V and edge set E that is embedded on a surface Σ of Euler genus $g = g(\Sigma)$. Euler's formula tells us that

$$|V| - |E| + |F| \geqslant 2 - g,$$

where F is the set of faces and with equality holding if and only if every face is a 2-cell. Therefore, if $|V| \geqslant 3$, then $|E| \leqslant 3|V| - 6 + 3g$. For $g \geqslant 1$, this implies that $\sigma(G) \leqslant H(g)$, that is, every subgraph of G has a vertex of degree at most $H(g) - 1$, where

$$H(g) = \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor.$$

Consequently, if $g \geqslant 1$, then

$$\omega(G) \leqslant \chi(G) \leqslant \chi^\ell(G) \leqslant \sigma(G) \leqslant H(g).$$

For every surface Σ distinct from the Klein bottle, the Heawood number $H(g)$ is, in fact, the maximum chromatic number of graphs embeddable on Σ where the maximum is attained by the complete graph on $H(g)$ vertices. This landmark result, that was conjectured by Heawood [10], is due to Ringel [16] and Ringel and Youngs [17]. Conversely, every graph with chromatic number $H(g)$ embedded on Σ contains a complete graph on $H(g)$ vertices as a subgraph. This result was proved by Dirac [5,7] for the torus and $g \geqslant 4$ and was proved by Albertson and Hutchinson [1] for $g = 1, 3$.

Franklin [8] proved that the colouring problem for the Klein bottle does not have the answer $H(2) = 7$ but 6. Furthermore, there are 6-chromatic graphs on the Klein bottle without a K_6. One example of such a graph is given in [1]. Brooks' theorem for the list-chromatic number implies that if G is a graph on the Klein bottle, then $\chi^\ell(G) \leqslant 6$. For graphs on the sphere the maximum chromatic number is 4; however, the maximum list-chromatic number is 5. The upper bound was proved by Thomassen [19] and the lower bound was proved by Voigt [21].

Let Σ be a surface of Euler genus $g \geqslant 0$. Then, for every integer $k \geqslant 1$, we have the familiar inequalities

$$\omega(k; \Sigma) \leqslant \chi(k; \Sigma) \leqslant \chi^\ell(k; \Sigma) \leqslant \sigma(k; \Sigma).$$

If Σ is distinct from the Klein bottle, then $\omega(1;\Sigma) = \sigma(1;\Sigma) = H(g)$. In [9] the following two theorems were proved. The first result establishes a lower bound on $\omega(k; \Sigma)$ that is approximately $(7k + \sqrt{24gk + k^2})/2$. The second result establishes an upper bound on $\sigma(k; \Sigma)$ that is asymptotic to this for fixed k and large g.

Theorem 1.1 (*Füredi et al. [9]*). *Let Σ be a surface of Euler genus $g \geqslant 1$ and let $k \geqslant 1$ be an integer. Then the following statements hold.*

(a) *If Σ is orientable, then $\omega(k; \Sigma) \geqslant k\, H(2\lfloor g/2k \rfloor)$.*
(b) *If Σ is non-orientable and $\lfloor g/k \rfloor \geqslant 3$, then $\omega(k; \Sigma) \geqslant k\, H(\lfloor g/k \rfloor)$.*

Theorem 1.2 (*Füredi et al. [9]*). *If Σ is a surface of Euler genus $g \geqslant 1$, then*

$$\sigma(k; \Sigma) \leqslant \left\lfloor \frac{7k + \sqrt{24kg + 49k^2 - 48k}}{2} \right\rfloor$$

for all integers $k \geqslant 1$.

For a given surface Σ of Euler genus $g \geqslant 0$, we define the number $H_2 = H_2(\Sigma)$ as follows: if Σ is orientable and $g = 12(2q + 1)^2$ for some integer $q \geqslant 0$, then

$$H_2 = 6 + \sqrt{12g} = 24q + 18.$$

If Σ is orientable and $g \equiv 2 \pmod 4$, then

$$H_2 = 7 + \lfloor \sqrt{12g} \rfloor.$$

In all other cases,

$$H_2 = 7 + \lfloor \sqrt{12g + 1} \rfloor.$$

The aim of this paper is to prove the following result.

Theorem 1.3. *Let Σ be a surface of Euler genus $g \geqslant 0$. If Σ is distinct from the non-orientable surfaces Π_4, then*

$$\omega(2; \Sigma) = \chi(2; \Sigma) = H_2(\Sigma).$$

Moreover,

$$\omega(2; \Pi_4) = \chi(2; \Pi_4) = H_2(\Pi_4) - 1 = 13.$$

1.2. Terminology

Concepts and notation not defined in this paper will be used as in standard textbooks.

A *graph* G is a pair consisting of a finite set $V(G)$ of *vertices* and a set $E(G)$ of 2-subsets of $V(G)$ called *edges*.

Let G be a graph. The *degree* $d_G(x)$ of a vertex $x \in V(G)$ is the number of edges in G that contain x. If $d_G(x) = r$ for every vertex $x \in V(G)$, then G is called *r-regular*. Furthermore, let $\delta(G)$ denote the minimum degree of G.

If H and G are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is said to be a *subgraph* of G. In this case we briefly write $H \subseteq G$.

Let G be a graph. For $X \subseteq V(G)$, the subgraph of G *induced* by X, written $G[X]$, is defined by $V(G[X]) = X$ and $E(G[X]) = \{e \in E(G) \mid e \subseteq X\}$. Let $G \setminus X$ denote $G[V(G) \setminus X]$.

For a graph G, a set $X \subseteq V(G)$ is a *clique* or an *independent set* of G if $G[X]$ is a complete graph or a graph without edges, respectively. The *clique number* of a graph G, denoted by $\omega(G)$, is the largest number m such that G contains a clique with m vertices. As usual, K_n denotes the complete graph on n vertices.

Consider a graph G and assign to each vertex x of G a set $L(x)$ of colours (positive integers). Such an assignment L of sets to vertices in G is referred to as a *list* for G. An *L-colouring* of G is a mapping c of $V(G)$ into the set of colours such that $c(x) \in L(x)$ for all $x \in V(G)$ and $c(x) \neq c(y)$ for each edge $\{x, y\} \in E(G)$. If G admits an L-colouring, then G is said to be *L-colourable*. When $L(x) = \{1, \dots, k\}$ for all $x \in V(G)$, the corresponding terms become *k-colouring* and *k-colourable*, respectively. G is said to be *k-list-colourable* if G is L-colourable for every list L of G satisfying $|L(x)| = k$ for all $x \in V(G)$.

The *chromatic number* of G denoted by $\chi(G)$ is the least number k such that G is k-colourable. The *list-chromatic number* of G denoted by $\chi^\ell(G)$ is the least number k such that G is k-list-colourable. The *colouring number* of a graph G, denoted by $\sigma(G)$, is defined by $\sigma(G) = 1 + \max_{H \subseteq G} \delta(H)$.

Let p be a graph parameter. A graph G is *p-critical* if $p(H) < p(G)$ for every proper subgraph H of G. For $p \in \{\omega, \chi, \chi^\ell, \sigma\}$, every graph G contains a p-critical subgraph H satisfying $p(H) = p(G)$. The importance of the notion of criticality is that problems for graphs may often be reduced to problems for critical graphs, whose structure is more restricted. Critical graphs were first defined and used by Dirac [4]. The next result is an extension of Brooks' theorem and was proved by Dirac [6] for the chromatic number and was proved by Kostochka and Stiebitz [12] for the list-chromatic number.

Theorem 1.4 (*Dirac [6] and Kostochka and Stiebitz [12]*). *Let $p \in \{\chi, \chi^\ell\}$. If G is a p-critical graph with $p(G) = k \geqslant 1$, then $\delta(G) \geqslant k - 1$ and, moreover,*

$$2|E(G)| \geqslant (k-1)|V(G)| + (k-3)$$

provided that $k \geqslant 4$ and $G \neq K_k$.

For a real number x, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the lower integer part of x and the upper integer part of x, respectively.

2. Embedding of graphs

To show that $\chi(2; \Sigma) \leqslant H_2(\Sigma)$ for every surface Σ of Euler genus $g \geqslant 0$ we need the Four-Colour Theorem in the case where $g = 0, 1, 2$. Apart from this fact the proof is easy if Σ is non-orientable. We only use Euler's formula and the result of Dirac [6]; see Theorem 1.4. However, for orientable surfaces the proof is much more involved and requires more information about embedded graphs.

First we need some notation. We define the *genus* $g(G)$ and the *non-orientable genus* $\tilde{g}(G)$ of a graph G as the minimum h and the minimum $\tilde{h}$, respectively, such that G has an embedding into the surface Σ_h, respectively, into the surface $\Pi_{\tilde{h}}$. Consequently, if $g(G) = h$, then G can be embedded on an orientable surface Σ iff $g(\Sigma) \geqslant 2h$. It is known, see [13], that $\tilde{g}(G) \leqslant 2g(G) + 1$ for every graph G. If $\tilde{g}(G) = 2g(G) + 1$, then G is said to be *orientably simple*. For instance, every planar graph G is orientably simple, since $g(G) = 0$ and $\tilde{g}(G) = 1$. Note that we consider trees as orientably simple graphs, although some authors exclude them.

An embedding of a graph G into a surface Σ is *cellular* if every face of G is homeomorphic to an open plane disk. Embeddings of G into $\Sigma_{g(G)}$ are called *minimum genus embeddings*. Similarly, a *non-orientable minimum genus embedding* of G is an embedding into $\Pi_{\tilde{g}(G)}$.

Theorem 2.1 (*Mohar and Thomassen [13] and Parson et al. [15]*). *Let G be a connected graph. Then the following statements hold.*

(a) *Every minimum genus embedding of G is cellular.*
(b) *If $\tilde{g}(G) \leqslant 2g(G)$, then every non-orientable minimum genus embedding of G is cellular.*

The genus of the complete graphs was established by Ringel and Youngs [17]. Battle et al. [2] proved that the genus of a graph is the sum of the genera of its blocks. Stahl and Beineke [18] proved a similar result for the non-orientable genus.

Theorem 2.2 (*Ringel [16] and Ringel and Youngs [17]*).

(a) $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ *for $n \geqslant 3$.*

(b) $\tilde{g}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$ *for $n \geqslant 3$ and $n \neq 7$.*

(c) *If Σ is a surface of Euler genus $g \geqslant 1$ distinct from the Klein bottle, then* $\omega(1;\Sigma) = \chi(1;\Sigma) = H(g)$. Moreover, $\omega(1;\Pi_2) = 6$.

Theorem 2.3 (*Battle et al. [2]*). *If G is the union of two connected graphs G_1 and G_2 which have exactly one vertex in common, then $g(G) = g(G_1) + g(G_2)$.*

Theorem 2.4 (*Stahl and Beineke [18]*). *If G is the union of two connected graphs G_1 and G_2 which have exactly one vertex in common, then $\tilde{g}(G) = \tilde{g}(G_1) + \tilde{g}(G_2) - \varepsilon$ where $\varepsilon = 0$ if neither G_1 nor G_2 is orientably simple, and $\varepsilon = 1$ otherwise.*

Theorem 2.5 (*Dirac [5] and Albertson and Hutchinson [1]*). *Let Σ be a surface of Euler genus $g \geqslant 1$. If G is an χ-critical graph embedded on Σ, then $\chi(G) \leqslant H(g)$ where equality holds if and only if G is a complete graph on $H(g)$ vertices.*

Theorem 2.6 (*Böhme et al. [3]*). *Let Σ be a surface of Euler genus g with $g \geqslant 1$ and $g \neq 3$. If G is an χ^ℓ-critical graph embedded on Σ, then $\chi(G) \leqslant H(g)$ where equality holds if and only if G is a complete graph on $H(g)$ vertices.*

If G and H are graphs, then the *complete join* $G + H$ is the graph obtained from the disjoint union of G and H by adding all edges between G and H. Clearly, $\chi(G + H) = \chi(G) + \chi(H)$. Moreover, if G and H are both χ-critical, then $G + H$ is χ-critical too.

Dirac [5] proved that for each fixed surface Σ and each integer $k \geqslant 7$ there are only finitely many χ-critical graphs with chromatic number k that can be embedded on Σ. The proof of this statement is mainly based on Euler's formula and Theorem 1.4 (see also [13]). The proof of the next result needs much more complicated tools and arguments.

Theorem 2.7 (*Thomassen [20]*). *There are precisely four χ-critical graphs with chromatic number six that can be embedded on the torus Σ_2, namely K_6, $K_3 + C_5$ (where C_5 is the 5-cycle), $K_2 + H_7$ (where H_7 is obtained from two complete graphs on four vertices by Hajós construction) and the graph T_{11} which is obtained from the 11-cycle $x_0x_1 \dots x_{10}x_0$ by adding all chords x_ix_{i+j} ($i = 0, \dots, 10$, $j = 2, 3$, all indices modulo 11).*

3. Proof of the upper bound

The following theorem immediately implies that $\chi(2;\Sigma) \leqslant H_2(\Sigma)$ for every surface Σ of Euler genus $g \geqslant 0$ and, moreover, that $\chi(2;\Pi_4) \leqslant H_2(\Pi_4) - 1 = 13$.

Theorem 3.1. *Let Σ be a surface of Euler genus $g \geqslant 0$ and let G be a graph embedded on Σ. Suppose that G is the edge disjoint union of two subgraphs G_1 and G_2. Then $\chi(G_1) + \chi(G_2) \leqslant H_2(\Sigma)$ for all Σ and, moreover, $\chi(G_1) + \chi(G_2) \leqslant 13$ provided that $\Sigma = \Pi_4$.*

Proof. Let Σ be a surface of Euler genus $g \geqslant 0$ and let G be a graph embedded on Σ where G is the edge disjoint union of two subgraphs G_1 and G_2. For the proof, we may assume that both graphs G_1 and G_2 are χ-critical. Let $k_i = \chi(G_i)$ for $i = 1, 2$ and $S = k_1 + k_2$. Suppose that $k_1 \leqslant k_2$ and

$$S = k_1 + k_2 \geqslant H_2 = H_2(\Sigma). \tag{1}$$

Then we have to show that $S = H_2$. Furthermore, we have to show that (1) leads to a contradiction if $\Sigma = \Pi_4$ (see Case C). Define

$$n_1 = |V(G_1) \setminus V(G_2)|,\ n_2 = |V(G_2) \setminus V(G_1)| \text{ and } n_{1,2} = |V(G_1) \cap V(G_2)|.$$

Then $n = n_1 + n_2 + n_{1,2}$ is the number of vertices of G and we may assume that $n_{1,2} \geqslant 1$. Otherwise, we can modify the embedding of the two graphs so that they share a vertex. Since $H_2 = H_2(\Sigma) \geqslant 8$ and $S \leqslant \chi(2; K_n) = n + 1$, we infer that $n \geqslant 7$. Furthermore, since G_i $(i = 1, 2)$ is χ-critical, we infer that G_i is connected. Consequently, G is connected and has $n \geqslant 7$ vertices. This implies, in particular, that every face of G has size at least 3.

Let $e = |E(G)|$, let f be the number of faces of G, and, moreover, let ℓ_G denote the maximum size of a face of G. Since every face of G has size at least 3, we have $\ell_G \geqslant 3$ and $2e \geqslant \ell_G + 3(f - 1)$. Since $n - e + f \geqslant 2 - g$ by Euler's formula, we then infer that

$$3n + 3g - 3 - \ell_G \geqslant e. \tag{2}$$

Case A: $g \in \{0, 1, 2\}$. Clearly, $k_1 \leqslant k_2 \leqslant H(g)$. Therefore, for $g = 0$, the Four-Colour Theorem implies that $S = k_1 + k_2 \leqslant 8 = H_2$ and we are done.

Now assume $g \in \{1, 2\}$. If $k_2 = H(g)$, then Theorem 2.5 implies that G_2 is a complete graph with $H(g)$ vertices. From Theorems 2.2 and 2.1 we conclude that the embedding of G_2 into Σ is cellular. Consequently, $V(G_2)$ is an independent set in G_1 and, moreover, the graph G_1' obtained from G_1 by identifying the vertices of G_2 to a single vertex is planar. Hence, by the Four-Colour Theorem, $k_1 = \chi(G_1) \leqslant \chi(G_1') \leqslant 4$ and, therefore, $S = k_1 + k_2 \leqslant 4 + H(g) \leqslant H_2$. Note that, for $g = 1$, we have $H(g) = 6$ and $H_2 = 10$ and, for $g = 2$, we have $H(g) = 7$ and $11 \leqslant H_2$.

If $k_2 \leqslant H(g) - 1$, then we argue as follows: for $g = 1$, this implies $S = k_1 + k_2 \leqslant 2H(1) - 2 = 10 = H_2$ and we are done. For $g = 2$ and $\Sigma = \Pi_2$, this implies $S = k_1 + k_2 \leqslant 2H(2) - 2 = 12 = H_2$ and we are done. Hence we need only consider the case $g = 2$ and $\Sigma = \Sigma_1$. Then $2H(2) - 2 = 12$ but $H_2 = H_2(\Sigma_1) = 11$. If $k_1 \leqslant 5$, then $S = k_1 + k_2 \leqslant 11 = H_2$ and we are also done. Otherwise, both graphs G_1 and G_2 are χ-critical with chromatic number 6. By Theorem 2.7, there are precisely four such graphs, that is, $G_1, G_2 \in \{L_1 = K_6, L_2 = K_3 + C_5, L_3 = K_2 + H_7, L_4 = T_{11}\}$ where $6|V(L_i)| - 2|E(L_i)| \leqslant 2$ for $i = 2, 3, 4$ and $6|V(L_1)| - 2|E(L_1)| \leqslant 6$. Then $g(G) = g(G_1) = g(G_2) = 1$ and, by Theorem 2.3, G_1 and G_2 have at least two vertices in common, that is, $n_{1,2} \geqslant 2$. Hence at least one of the two graphs G_1 and G_2 is not a K_6 and, therefore,

$$6n - 2e = 6|V(G_1)| - 2|E(G_1)| + 6|V(G_2)| - 2|E(G_2)| - 6n_{1,2} \leqslant -4,$$

a contradiction to (2). This settles the case $g = 2$ and $\Sigma = \Sigma_1$.

Case B: $g \geqslant 3$. Consider first the case where $k_1 \leqslant 6$. For $g \geqslant 12$, this implies, by an easy calculation, that

$$S = k_1 + k_2 \leqslant H(g) + 6 \leqslant \frac{7 + \sqrt{24g + 1}}{2} + 6 \leqslant 6 + \sqrt{12g} \leqslant H_2.$$

Table 1
The functions $H(g)$ and $H_2(\Sigma)$ for a surface Σ of Euler genus g

g	0	1	2	3	4	5	6	7	8	9	10	11	12
$H(g)$	4	6	7	7	8	9	9	10	10	10	11	11	12
$H_2(\Sigma)$	8	10	11/12	13	14	14	15	16	16	17	17/18	18	18/19

If $3 \leqslant g \leqslant 11$ and $g \neq 5$, then it can be verified manually that $S = k_1 + k_2 \leqslant H(g) + 6 \leqslant H_2$ (see Table 1). Eventually, suppose $g = 5$. Then $H(g) = 9$ and $H_2 = 14$. If $k_2 \leqslant 8$, then $S = k_1 + k_2 \leqslant 14 = H_2$ and we are done. Otherwise, $k_2 = 9$ and from Theorems 2.5 and 2.1 we infer that the embedding of $G_2 = K_9$ into $\Sigma = \Pi_5$ is cellular. Consequently, $V(G_2)$ is an independent set in G_1 and, moreover, the graph G_1' obtained from G_1 by identifying the vertices of G_2 to a single vertex is planar. Hence, $k_1 = \chi(G_1) \leqslant \chi(G_1') \leqslant 4$ and, therefore, $S = k_1 + k_2 \leqslant 13 \leqslant H_2$. This completes the proof in case $k_1 \leqslant 6$.

Consider next the main case where $k_2 \geqslant k_1 \geqslant 7$. For $i = 1, 2$; let $d_i = 2|E(G_i)| - (k_i - 1)|V(G_i)|$. Theorem 1.4 implies that $d_i \geqslant k_i - 3 \geqslant 4$ provided that $G_i \neq K_{k_i}$. Clearly, if $G_i = K_{k_i}$, then $d_i = 0$. Since $|V(G_i)| = n_i + n_{1,2} \geqslant k_i$ and $k_i \geqslant 7$, we obtain

$$2|E(G_i)| = (k_i - 1)(n_i + n_{1,2}) + d_i \geqslant (k_i - 7)k_i + d_i + 6(n_i + n_{1,2}).$$

Since $e = |E(G_1)| + |E(G_2)|$ and $n = n_1 + n_2 + n_{1,2}$, we then infer from (2) that

$$6g - 6 - 2\ell_G \geqslant (k_1 - 7)k_1 + (k_2 - 7)k_2 + (d_1 + d_2) + 6n_{1,2}. \tag{3}$$

Now, we distinguish two subcases.

Subcase 1: $n_{1,2} \geqslant 2$. Then G_1 or G_2 is not a complete graph and Theorem 1.4 implies that $d_1 + d_2 \geqslant \min(k_1, k_2) - 3 \geqslant 4$. From (3) it then follows that

$$\begin{aligned} 6g - 12 &\geqslant (k_1 - 7)k_1 + (k_2 - 7)k_2 + 16 \\ &= (k_1 - \tfrac{7}{2})^2 + (k_2 - \tfrac{7}{2})^2 - \tfrac{17}{2} \\ &\geqslant \tfrac{1}{2}(k_1 + k_2 - 7)^2 - \tfrac{17}{2}. \end{aligned}$$

Consequently, $S = k_1 + k_2 \leqslant 7 + \lfloor\sqrt{12g - 7}\rfloor \leqslant H_2$. Note that in case $g = 12(2q + 1)^2$ we have $7 + \lfloor\sqrt{12g - 7}\rfloor = 6 + 12(2q + 1) = H_2$. This settles the case $n_{1,2} \geqslant 2$.

Subcase 2: $n_{1,2} = 1$. Then $\ell_G \geqslant 6$ and $d_1 + d_2 \geqslant 0$. Hence, (3) implies that

$$6g - 24 \geqslant (k_1 - 7)k_1 + (k_2 - 7)k_2. \tag{4}$$

Subcase 2.1: *Both G_1 and G_2 are complete graphs.* Then $G_i = K_{k_i}$ for $i = 1, 2$. Let a, b be real numbers satisfying $1 \leqslant a, b \leqslant 7$ and $a + b = 7$. Then for $M = (k_1 - 7)k_1 + (k_2 - 7)k_2$ we obtain

$$\begin{aligned} M &= (k_1 - a)^2 + (k_2 - b)^2 - (a^2 + b^2) + (2a - 7)k_1 + (2b - 7)k_2 \\ &\geqslant \tfrac{1}{2}(k_1 + k_2 - 7)^2 - (a^2 + b^2) + (2a - 7)k_1 + (2b - 7)k_2. \end{aligned}$$

By (4), this implies that

$$S = k_1 + k_2 \leqslant 7 + \left\lfloor \sqrt{12g - 48 + 2(a^2 + b^2) - 2(2a - 7)k_1 - 2(2b - 7)k_2} \right\rfloor. \tag{5}$$

Then, in case of $(a,b)=\left(\frac{7}{2},\frac{7}{2}\right)$, we obtain that

$$S\leqslant 7+\left\lfloor\sqrt{12g+1}\right\rfloor. \tag{6}$$

Consequently, $S\leqslant H_2$ or Σ is orientable and either $g=12(2q+1)^2$ for some integer $q\geqslant 0$ or $g\equiv 2 \pmod 4$.

First consider the case where Σ is orientable and $g=12(2q+1)^2$. Then $H_2=24q+18$ and, by (6), $S\leqslant 24q+19$. Suppose $S\geqslant H_2+1$. Then $S=24q+19$ and we arrive at a contradiction as follows: if $k_2=k_1+1$, then $k_1=12q+9$ and $k_2=12q+10$. Therefore, Theorem 2.2 implies that

$$\begin{aligned}g(G_1)=g(K_{k_1})&=\left\lceil\frac{(6+12q)(5+12q)}{12}\right\rceil\\&=\frac{(1+2q)(5+12q)+1}{2}\end{aligned}$$

and

$$\begin{aligned}g(G_2)=g(K_{k_2})&=\left\lceil\frac{(7+12q)(6+12q)}{12}\right\rceil\\&=\frac{(1+2q)(7+12q)+1}{2}.\end{aligned}$$

Consequently, by Theorem 2.3,

$$2g(G)=2g(G_1)+2g(G_2)=(1+2q)(12+24q)+2>12(1+2q)^2=g,$$

a contradiction. Otherwise, $k_2\geqslant k_1+2$ and, if we apply (5) with $(a,b)=\left(\frac{5}{2},\frac{9}{2}\right)$, then we obtain

$$S\leqslant 7+\left\lfloor\sqrt{12g+5+4(k_1-k_2)}\right\rfloor\leqslant 7+\left\lfloor\sqrt{12g-3}\right\rfloor\leqslant H_2,$$

a contradiction. This proves $S\leqslant H_2$.

Eventually consider the case where Σ is orientable and $g\equiv 2 \pmod 4$. If $2g(G)=g'<g$, then (6) implies that $S\leqslant 7+\lfloor\sqrt{12g'+1}\rfloor\leqslant 7+\lfloor\sqrt{12g}\rfloor=H_2$ and we are done. Otherwise, $2g(G)=g\equiv 2\pmod 4$ and Theorem 2.3 implies that $g(G_1)\neq g(G_2)$ and, therefore, $k_1\neq k_2$. But then $k_2\geqslant k_1+1$ and if we apply (5) with $(a,b)=(3,4)$, then we obtain

$$S\leqslant 7+\left\lfloor\sqrt{12g+2+2(k_1-k_2)}\right\rfloor\leqslant 7+\left\lfloor\sqrt{12g}\right\rfloor=H_2$$

and we are also done. This completes the proof for Subcase 2.1.

Subcase 2.2: G_1 *or* G_2 *is not complete.* First consider the case where Σ is orientable. For $i=1,2$, let h_i be the orientable genus of G_i, i.e., h_i is the smallest integer such that G_i can be embedded on Σ_{h_i}. Let K^i denote the complete graph with $H(2h_i)$ vertices and, moreover, let K denote the graph obtained from the disjoint union $K^1\cup K^2$ by identifying a vertex of K^1 with some vertex of K^2. By Theorem 2.2, K^i can be embedded on Σ_{h_i} for

$i = 1, 2$, and, therefore, K can be embedded on Σ (see Theorem 2.3). Since G_1 or G_2 is not complete, Theorem 2.5 implies that $S = \chi(G_1)+\chi(G_2) \leqslant \chi(K^1)+\chi(K^2)-1$. Furthermore, by Subcase 2.1, $\chi(K^1) + \chi(K^2) \leqslant H_2 = H_2(\Sigma)$. Hence $S \leqslant H_2 - 1$, a contradiction to (1).

Now consider the case that Σ is non-orientable, say $\Sigma = \Pi_h$ for some positive integer h. For $i = 1, 2$, let h_i be the non-orientable genus of G_i, i.e., h_i is the smallest positive integer such that G_i can be embedded on Π_{h_i}. Since $k_i = \chi(G_i) \geqslant 7$, we have $h_i \geqslant 3$. Let K^i denote the complete graph with $H(h_i)$ vertices and, moreover, let K denote the graph obtained from the disjoint union $K^1 \cup K^2$ by identifying a vertex of K^1 with some vertex of K^2. By Theorem 2.2, K^i can be embedded on Π_{h_i} for $i = 1, 2$.

If neither G_1 nor G_2 is orientably simple, then we infer from Theorem 2.4 that $h \geqslant h_1+h_2$ and that K can be embedded on $\Sigma = \Pi_h$. Since G_1 or G_2 is not complete, Theorem 2.5 implies that $S = \chi(G_1) + \chi(G_2) \leqslant \chi(K^1) + \chi(K^2) - 1$. Furthermore, by Subcase 2.1, $\chi(K^1) + \chi(K^2) \leqslant H_2 = H_2(\Sigma)$. Hence $S \leqslant H_2 - 1$, a contradiction to (1).

If G_1 or G_2 is orientably simple, then we argue as follows: Theorem 2.4 implies that $h \geqslant h_1 + h_2 - 1$. If $h \geqslant h_1 + h_2$, then we arrive at a contradiction in the same way as above. Otherwise, $h = h_1 + h_2 - 1$ and, by Theorem 2.4, K can be embedded on Π_{h+1}. Then, by Theorem 2.5 and Subcase 2.1, we have $S = \chi(G_1) + \chi(G_2) \leqslant \chi(K^1) + \chi(K^2) - 1 \leqslant H_2(\Pi_{h+1}) - 1 \leqslant H_2(\Pi_h) = H_2(\Sigma)$. The last inequality follows from the fact that $h = h_1 + h_2 \geqslant 6$ and that therefore $\sqrt{12(h + 1) + 1} - 1 \leqslant \sqrt{12h + 1}$.

Case C: $\Sigma = \Pi_4$. Then we have to show that the assumption $S = k_1 + k_2 \geqslant H_2(\Sigma) = 14$ leads to a contradiction. Since $g(\Pi_4) = 4$ and $H(4) = 8$, we have $k_1 \leqslant k_2 \leqslant 8$.

Consider first the case where $k_2 = 8$. Then Theorem 2.5 implies that $G_2 = K_8$. From Theorem 2.2 and Theorem 2.1 we conclude that the embedding of G_2 into Σ is cellular. Consequently, $V(G_2)$ is an independent set in G_1 and, moreover, the graph G_1' obtained from G_1 by identifying the vertices of G_2 to a single vertex is planar. Hence, by the Four-Colour Theorem, $k_1 = \chi(G_1) \leqslant \chi(G_1') \leqslant 4$ and, therefore, $S = k_1 + k_2 \leqslant 12$, a contradiction.

Consider next the case where $k_2 \leqslant 7$. Since $S = k_1 + k_2 \geqslant 14$ and $k_1 \leqslant k_2$, this implies that $k_1 = k_2 = 7$. As in Case B, let $d_i = 2|E(G_i)| - 6|V(G_i)|$. Then, see (3), we have

$$18 - 2\ell_G \geqslant (d_1 + d_2) + 6n_{1,2}. \tag{7}$$

If $n_{1,2} \geqslant 2$, then G_1 or G_2 is not a complete graph and, therefore, Theorem 1.4 implies that $d_1 + d_2 \geqslant 4$. Since $\ell_G \geqslant 3$, this gives a contradiction to (7).

Otherwise, $n_{1,2} = 1$ and, therefore, $\ell_G \geqslant 6$. Then, by (7), it follows that $d_1 + d_2 = 0$. Consequently, by Theorem 1.4, $G_1 = G_2 = K_7$. By Theorems 2.2 and 2.4, this implies that $\tilde{g}(G) \geqslant 2\tilde{g}(K_7) - 1 = 5$, a contradiction to the assumption that G is embedded on Π_4. This contradiction completes the proof for Case C.

Hence Theorem 3.1 is proved. □

4. Proof of the lower bound

Let K and K' be two complete graphs. Then the graph obtained from disjoint copies of K and K' by identifying a vertex of K with a vertex of K' is denoted by $K * K'$.

The aim of this section is to show that $\omega(2;\Sigma) \geqslant H_2(\Sigma)$ for every surface Σ distinct from the non-orientable surfaces Π_4. The proof of this inequality is mainly based on the following proposition.

Proposition 4.1. *Let Σ be a surface of Euler genus $g \geqslant 0$. Then the following statements hold.*

(a) *If Σ is orientable and $g = g_1 + g_2$ where $g_1, g_2 \geqslant 0$ are even integers, then $\omega(2;\Sigma) \geqslant H(g_1) + H(g_2)$.*
(b) *If Σ is non-orientable and $g = g_1 + g_2$ where $g_1, g_2 \geqslant 0$ are integers with $g_1, g_2 \neq 2$, then $\omega(2;\Sigma) \geqslant H(g_1) + H(g_2)$.*

Proof. For the proof of (a), assume that Σ is an orientable surface of Euler genus $g \geqslant 0$ and that $g_1, g_2 \geqslant 0$ are even integers satisfying $g = g_1 + g_2$. Then Theorem 2.2 implies that, for $i = 1, 2$, there is a complete graph K^i with $H(g_i)$ vertices that can be embedded on an orientable surface of Euler genus g_i. Since $g_1 + g_2 = g$, it then follows from Theorem 2.3 that the graph $K^1 * K^2$ can be embedded on Σ. Consequently, $\omega(2;\Sigma) \geqslant H(g_1) + H(g_2)$.

For the proof of (b), assume that Σ is a non-orientable surface of Euler genus $g \geqslant 1$ and that $g_1, g_2 \geqslant 0$ are integers satisfying $g_1, g_2 \neq 2$ and $g = g_1 + g_2$. Then Theorem 2.2 implies that, for $i = 1, 2$, there is a complete graph K^i with $H(g_i)$ vertices that can be embedded on a non-orientable surface of Euler genus g_i or, in case of $g_i = 0$, on the sphere Σ_0. Since $g_1 + g_2 = g$, it then follows from Theorem 2.4 that the graph $K^1 * K^2$ can be embedded on Σ. Consequently, $\omega(2;\Sigma) \geqslant H(g_1) + H(g_2)$. $\square$

Therefore, our aim is to show that for every surface $\Sigma \neq \Pi_4$ there is a certain pair (g_1, g_2) of two integers such that $H(g_1) + H(g_2) \geqslant H_2(\Sigma)$. For an integer $s \geqslant 1$, let

$$a(s) = \left\lceil \frac{s^2 - 1}{24} \right\rceil$$

and, moreover, let

$$d(s) = a(s+1) - a(s).$$

Clearly, for every integer $g \geqslant a(s)$, we have

$$H(g) = \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor \geqslant \left\lfloor \frac{7 + s}{2} \right\rfloor.$$

The following proposition establishes some useful properties of the function $a = a(s)$.

Proposition 4.2. *Let $s \geqslant 7$ be an integer. Then statements* (a), (b) *and* (c) *hold provided that s is even and statements* (d), (e) *and* (f) *hold provided that s is odd.*

(a) $a(s) - d(s) \geqslant \left\lceil \frac{s^2-1}{24} - \frac{1}{2} \right\rceil - d(s) \geqslant a(s-1)$.
(b) *If $a(s) \equiv 0 \pmod 2$ and $d(s) \equiv 1 \pmod 2$, then $a(s) = 6(2q+1)^2$ for some integer $q \geqslant 0$ or $a(s) - d(s) - 1 \geqslant a(s-1)$.*

(c) *If* $a(s) \equiv 1 \pmod 2$ *and* $d(s) \equiv 0 \pmod 2$, *then there is an odd integer* $d' \geqslant 1$ *and an integer* $k \in \{1, 3\}$ *such that* $a(s+k) - a(s) \leqslant d' \leqslant a(s) - a(s-k)$.

(d) *If* $g' = \left\lceil \frac{s^2-1}{24} - \frac{1}{2} \right\rceil < a(s)$, *then there is an integer* $d' \geqslant 0$ *such that* $g' - d' \geqslant a(s-2)$ *and* $g' + d' + 1 \geqslant a(s+2)$.

(e) *If* $a(s) = \left\lceil \frac{s^2}{24} \right\rceil$, *then there is an odd integer* $d' \geqslant 1$ *such that* $a(s+2) - a(s) \leqslant d' \leqslant a(s) - a(s-2)$.

(f) $a(s) - a(s-2) \geqslant 1$.

Proof. For the proof of statements (a), (b) and (c), assume that $s = 2t$ for some integer $t \geqslant 4$. Then there are unique integers p, r satisfying $t = 6p + r$ and $0 \leqslant r \leqslant 5$. This implies that $s = 12p + 2r$ and $\frac{s^2}{24} = 6p^2 + 2pr + \frac{r^2}{6}$. Furthermore, we have

$$\begin{aligned}
a(s) &= 6p^2 + 2pr + \left\lceil \frac{4r^2 - 1}{24} \right\rceil, \\
a(s-1) &= 6p^2 + 2pr - p + \left\lceil \frac{r^2 - r}{6} \right\rceil, \\
a(s+1) &= 6p^2 + 2pr + p + \left\lceil \frac{r^2 + r}{6} \right\rceil, \\
a(s-3) &= 6p^2 + 2pr - 3p + \left\lceil \frac{r^2}{6} - \frac{r}{2} + \frac{1}{3} \right\rceil, \\
a(s+3) &= 6p^2 + 2pr + 3p + \left\lceil \frac{r^2}{6} + \frac{r}{2} + \frac{1}{3} \right\rceil
\end{aligned}$$

and,

$$\left\lceil \frac{s^2 - 1}{24} - \frac{1}{2} \right\rceil = 6p^2 + 2pr + \left\lceil \frac{4r^2 - 13}{24} \right\rceil.$$

Since $d(s) = a(s+1) - a(s)$, the inequality

$$\left\lceil \frac{s^2 - 1}{24} - \frac{1}{2} \right\rceil - d(s) \geqslant a(s-1)$$

is equivalent to

$$\left\lceil \frac{4r^2 - 13}{24} \right\rceil + \left\lceil \frac{4r^2 - 1}{24} \right\rceil \geqslant \left\lceil \frac{r^2 - r}{6} \right\rceil + \left\lceil \frac{r^2 + r}{6} \right\rceil.$$

The last inequality holds for $r = 0, 1, 2, 3, 4, 5$ (see Table 2). Hence (a) is proved.

Clearly, $a(s) \equiv 0 \pmod 2$ if and only if $r \in \{0, 3\}$. Moreover,

$$d(s) = p + \left\lceil \frac{r^2 + r}{6} \right\rceil - \left\lceil \frac{4r^2 - 1}{24} \right\rceil.$$

For the proof of (b), assume that $a(s)$ is even and $d(s)$ is odd. Then $r \in \{0, 3\}$ and $d(s) = p$. If $r = 0$, then $a(s) = 6p^2$ where p is odd. If $r = 3$, then we have $a(s) = 6p^2 + 6p + 2$

Table 2
Several ceiling functions $f(r)$ for $r = 0, 1, 2, 3, 4, 5$

r	0	1	2	3	4	5
$\left\lceil \frac{r^2-r}{6} \right\rceil$	0	0	1	1	2	4
$\left\lceil \frac{4r^2-13}{24} \right\rceil$	0	0	1	1	3	4
$\left\lceil \frac{4r^2-1}{24} \right\rceil$	0	1	1	2	3	5
$\left\lceil \frac{r^2+r}{6} \right\rceil$	0	1	1	2	4	5
$\left\lceil \frac{r^2+r}{6} + \frac{1}{24} \right\rceil$	1	1	2	3	4	6
$\left\lceil \frac{r^2+r}{6} - \frac{1}{2} \right\rceil$	0	0	1	2	3	5

and $a(s-1) = 6p^2 + 5p + 1$. This implies $a(s) - d(s) - 1 = a(s-1)$. Hence (b) is proved.

For the proof of (c), assume that $a(s)$ is odd and $d(s)$ is even. Then $r \in \{1, 2, 4, 5\}$. If $r \in \{1, 5\}$, then $d(s) = p$ and we have

$$a(s+1) - a(s) = p \leqslant p + 1 = p + \left\lceil \frac{4r^2-1}{24} \right\rceil - \left\lceil \frac{r^2-r}{6} \right\rceil = a(s) - a(s-1).$$

If $r = 2$, then $d(s) = p$ and $a(s+3) - a(s) = 3p + 1 = a(s) - a(s-3)$. If $r = 4$, then $d(s) = p + 1$ and $a(s+3) - a(s) = 3p + 2 = a(s) - a(s-3)$. Hence (c) is proved.

For the proof of statements (d), (e) and (f), assume that $s = 2t + 1$ for some integer $t \geqslant 3$. Then there are unique integers p, r satisfying $t = 6p + r$ and $0 \leqslant r \leqslant 5$. This implies that $s = 12p + 2r + 1$ and $\frac{s^2}{24} = 6p^2 + 2pr + p + \frac{r^2+r}{6} + \frac{1}{24}$. Furthermore, we have

$$\begin{aligned}
a(s) &= 6p^2 + 2pr + p + \left\lceil \frac{r^2+r}{6} \right\rceil, \\
a(s-2) &= 6p^2 + 2pr - p + \left\lceil \frac{r^2-r}{6} \right\rceil, \\
a(s+2) &= 6p^2 + 2pr + 3p + \left\lceil \frac{r^2}{6} + \frac{r}{2} + \frac{1}{3} \right\rceil, \\
\left\lceil \frac{s^2}{24} \right\rceil &= 6p^2 + 2pr + p + \left\lceil \frac{r^2+r}{6} + \frac{1}{24} \right\rceil
\end{aligned}$$

and,

$$\left\lceil \frac{s^2-1}{24} - \frac{1}{2} \right\rceil = 6p^2 + 2pr + p + \left\lceil \frac{r^2+r}{6} - \frac{1}{2} \right\rceil.$$

For the proof of (d), assume that $g' = \left\lceil \frac{s^2-1}{24} - \frac{1}{2} \right\rceil < a(s)$. Then, $\left\lceil \frac{r^2+r}{6} - \frac{1}{2} \right\rceil < \left\lceil \frac{r^2+r}{6} \right\rceil$. Consequently, see Table 2, we have $r \in \{1, 4\}$. Since

$$g' - a(s-2) = 2p + \left\lceil \frac{r^2+r}{6} - \frac{1}{2} \right\rceil - \left\lceil \frac{r^2-r}{6} \right\rceil$$

and

$$a(s+2) - g' = 2p + \left\lceil \frac{r^2}{6} + \frac{r}{2} + \frac{1}{3} \right\rceil - \left\lceil \frac{r^2+r}{6} - \frac{1}{2} \right\rceil,$$

this implies that $a(s+2) - g' = g' - a(s-2) + 1$. Hence (d) is proved.

For the proof of (e), assume that $\left\lceil \frac{s^2}{24} \right\rceil = a(s)$. Then $\left\lceil \frac{r^2+r}{6} + \frac{1}{24} \right\rceil = \left\lceil \frac{r^2+r}{6} \right\rceil$. Consequently, see Table 2, we have $r \in \{1, 4\}$. Since

$$a(s) - a(s-2) = 2p + \left\lceil \frac{r^2+r}{6} \right\rceil - \left\lceil \frac{r^2-r}{6} \right\rceil$$

and

$$a(s+2) - a(s) = 2p + \left\lceil \frac{r^2}{6} + \frac{r}{2} + \frac{1}{3} \right\rceil - \left\lceil \frac{r^2+r}{6} \right\rceil,$$

this implies that $a(s) - a(s-2) \geqslant 2p + 1 \geqslant a(s+2) - a(s)$. Hence (e) is proved.

Since $s \geqslant 7$, we have $p \geqslant 1$ or $r \geqslant 3$ and, therefore,

$$a(s) - a(s-2) = 2p + \left\lceil \frac{r^2+r}{6} \right\rceil - \left\lceil \frac{r^2-r}{6} \right\rceil \geqslant 1.$$

This proves (f). $\square$

That $\omega(2;\Sigma) \geqslant H_2(\Sigma)$ for every non-orientable surface $\Sigma \neq \Pi_3, \Pi_4, \Pi_7$ follows from Proposition 4.1 and the next proposition.

Proposition 4.3. *Let $g \geqslant 1$ be an integer where $g \notin \{3, 4, 7\}$. Then there are two integers $g_1, g_2 \geqslant 0$ satisfying $g_1, g_2 \neq 2$, $g = g_1 + g_2$ and $H(g_1) + H(g_2) \geqslant 7 + \lfloor \sqrt{12g+1} \rfloor$.*

Proof. Let $g \geqslant 1$ be an integer where $g \notin \{3, 4, 7\}$ and let $s = \lfloor \sqrt{12g+1} \rfloor$. A pair (g_1, g_2) of integers is called *feasible* if $g_2 \geqslant g_1 \geqslant 0$, $g_1, g_2 \neq 2$, $g = g_1 + g_2$ and $H(g_1) + H(g_2) \geqslant 7 + s$. We have to show that there is a feasible pair (g_1, g_2). With the help of Table 1 it is easy to check that the pair $(0, 1)$ is feasible for $g = 1$, the pair $(1, 1)$ is feasible for $g = 2$, the pair

$(1,4)$ is feasible for $g=5$, and the pair $(1,5)$ is feasible for $g=6$. Hence in what follows we assume $g \geqslant 8$.

Case 1: *g is even, say,* $g=2g'$. Since $g \geqslant 8$, we have $s=\lfloor\sqrt{12g+1}\rfloor \geqslant 9$ and $a(s) \leqslant g' < a(s+1)$.

If s is odd, this implies that $H(g') \geqslant \frac{7+s}{2}$ and, therefore, $(g_1,g_2)=(g',g')$ is a feasible pair.

If s is even, then, by Proposition 4.2(a), it follows that $g_1=g'-d(s) \geqslant a(s)-d(s) \geqslant a(s-1)$ and, hence, $H(g_1) \geqslant \frac{6+s}{2}$. Furthermore, $g_2=g'+d(s) \geqslant a(s+1)$ and, hence, $H(g_2) \geqslant \frac{8+s}{2}$. Consequently, $H(g_1)+H(g_2) \geqslant 7+s$. Since $s \geqslant 9$, we have $g_2 \geqslant g_1 \geqslant a(8)=3$. Therefore, (g_1,g_2) is a feasible pair.

Case 2: *g is odd, say* $g=2g'+1$. Then $g \geqslant 9$ and, therefore, we have $s=\lfloor\sqrt{12g+1}\rfloor \geqslant 10$ and

$$\left\lceil \frac{s^2-1}{24}-\frac{1}{2}\right\rceil \leqslant g' < \left\lceil \frac{(s+1)^2-1}{24}-\frac{1}{2}\right\rceil.$$

Consider first the case where s is even. Then from Proposition 4.2(a) it follows that $g_1=g'-d(s) \geqslant a(s-1)$ and, hence, $H(g_1) \geqslant \frac{6+s}{2}$. Furthermore, it follows that $g_2=g'+d(s)+1 \geqslant a(s+1)$ and, hence, $H(g_2) \geqslant \frac{8+s}{2}$. Consequently, $g_1+g_2=g$, $H(g_1)+H(g_2) \geqslant 7+s$ and $g_2 \geqslant g_1 \geqslant a(9)=4$. Hence (g_1,g_2) is a feasible pair.

Consider next the case where s is odd. If $g' \geqslant a(s)$, then $H(g'+1) \geqslant H(g') \geqslant \frac{7+s}{2}$ and, therefore, $(g',g'+1)$ is a feasible pair. Otherwise, $g'=\left\lceil \frac{s^2-1}{24}-\frac{1}{2}\right\rceil < a(s)$ and, by Proposition 4.2(d), there is an integer $d' \geqslant 0$ such that $g_1=g'-d' \geqslant a(s-2)$ and $g_2=g'+d+1' \geqslant a(s+2)$. Consequently, $g_1+g_2=g$, $H(g_1)+H(g_2) \geqslant \frac{5+s}{2}+\frac{9+s}{2}=7+s$ and $g_2 \geqslant g_1 \geqslant a(8)=3$. Hence (g_1,g_2) is a feasible pair. □

For the non-orientable surfaces $\Sigma=\Pi_3,\Pi_4,\Pi_7$, we need a special argument. By a result of Franklin [8], it is known that $\tilde{g}(K_7)=3$. By Theorem 2.2, we have $\tilde{g}(K_6)=1$, $\tilde{g}(K_9)=5$, and, $g(K_7)=1$. This implies, in particular, that K_7 is orientably simple. Therefore, by Theorem 2.4, we have $\tilde{g}(K_6 * K_7)=3$ and $\tilde{g}(K_7 * K_9)=7$. Consequently, $\omega(2;\Pi_3) \geqslant 13 = H_2(\Pi_3)$, $\omega(2;\Pi_4) \geqslant 13 = H_2(\Pi_4)-1$ and $\omega(\Pi_7) \geqslant 16 = H(\Pi_7)$.

For an even number $g \geqslant 0$, let

$$F(g)=\begin{cases} 6+\sqrt{12g} & \text{if } g=12(2q+1)^2,\\ 7+\lfloor\sqrt{12g}\rfloor & \text{if } g\equiv 2 \pmod 4,\\ 7+\lfloor\sqrt{12g+1}\rfloor & \text{otherwise.}\end{cases}$$

Consider an orientable surface Σ with Euler genus g. Then g is even and $H_2(\Sigma)=F(g)$. That $\omega_2(2;\Sigma) \geqslant H_2(\Sigma)$ follows from Proposition 4.1 and the next proposition.

Proposition 4.4. *Let* $g \geqslant 0$ *be an even number. Then there are even numbers* $g_1, g_2 \geqslant 0$ *such that* $g=g_1+g_2$ *and* $H(g_1)+H(g_2) \geqslant F(g)$.

Proof. A pair (g_1, g_2) of even numbers is called *feasible* if $g = g_1 + g_2$ and $H(g_1) + H(g_2) \geqslant F(g)$. We have to show that there is a feasible pair.

Clearly, the pair $(0, 0)$ is feasible for $g = 0$ and the pair $(0,2)$ is feasible for $g = 2$. Now assume $g \geqslant 4$. Then $g = 2g'$ for some integer $g' \geqslant 2$. Let $s = \left\lfloor \sqrt{12g+1} \right\rfloor$. Then $s \geqslant 7$ and $a(s) \leqslant g' < a(s+1)$. Moreover, $6 + s \leqslant F(g) \leqslant 7 + s$.

Case 1: g' *is even*. If s is odd, then $H(g') \geqslant \frac{7+s}{2}$ and, therefore, $H(g') + H(g') \geqslant 7 + s \geqslant F(g)$. Consequently, (g', g') is a feasible pair.

If s is even, then we argue as follows: first consider the case where $d(s) = a(s+1) - a(s) \equiv 0 \pmod 2$. Then, by Proposition 4.2(a), it follows that $g_1 = g' - d(s) \geqslant a(s-1)$ and, hence, $H(g_1) \geqslant \frac{6+s}{2}$. Furthermore, $g_2 = g' + d(s) \geqslant a(s+1)$ and, hence, $H(g_2) \geqslant \frac{8+s}{2}$. Consequently, $H(g_1) + H(g_2) \geqslant 7 + s \geqslant F(g)$. Since both numbers g_1 and g_2 are even, this implies that (g_1, g_2) is a feasible pair.

Now consider the case where $d(s) = a(s+1) - a(s) \equiv 1 \pmod 2$. If $g' \geqslant a(s) + 1$, then $g_1 = g' - (d(s) - 1) \geqslant a(s-1)$ and $g_2 = g' + (d(s) - 1) \geqslant a(s+1)$ and, clearly, (g_1, g_2) is a feasible pair. Otherwise, $g' = a(s)$ and, by Proposition 4.2(b), $g = 2a(s) = 12(2q+1)^2$ or $g_1 = g' - (d(s)+1) \geqslant a(s-1)$. In the first case (g', g') is a feasible pair, since $H(g') + H(g') \geqslant 2\left\lfloor \frac{7+s}{2} \right\rfloor = 6 + s = F(g)$. In the second case, $g_2 = g' + (d(s)+1) \geqslant a(s+1)$ and, therefore, $H(g_1) + H(g_2) = 7 + s \geqslant F(g)$ and (g_1, g_2) is a feasible pair.

Case 2: g' *is odd*. Then $g \equiv 2 \pmod 4$ and, therefore, $F(g) = 7 + \left\lfloor \sqrt{12g} \right\rfloor$.

Consider first the case where s is even. If $d(s) \equiv 1 \pmod 2$, then, by Proposition 4.2(a), $g_1 = g' - d(s) \geqslant a(s-1)$ and, hence, $H(g_1) \geqslant \frac{6+s}{2}$. Furthermore, $g_2 = g' + d(s) \geqslant a(s+1)$ and, hence, $H(g_2) \geqslant \frac{8+s}{2}$. Consequently, $H(g_1) + H(g_2) \geqslant 7 + s \geqslant F(g)$. Since both numbers g_1 and g_2 are even, this implies that (g_1, g_2) is a feasible pair.

If $d(s) \equiv 0 \bmod 2$, then we argue as follows: if $g' \geqslant a(s) + 1$, then $g_1 = g' - (d(s) - 1) \geqslant a(s-1)$ and $g_2 = g' + (d(s) - 1) \geqslant a(s+1)$ and, clearly, (g_1, g_2) is a feasible pair. Otherwise, $g' = a(s)$ and, by Proposition 4.2(c), there is an odd integer $d' \geqslant 1$ and an integer $k \in \{1, 3\}$ such that $g_1 = g' - d' = a(s) - d' \geqslant a(s-k)$ and $g_2 = g' + d' = a(s) + d' \geqslant a(s+k)$. Then $H(g_1) + H(g_2) \geqslant \frac{7+s-k}{2} + \frac{7+s+k}{2} = 7 + s \geqslant F(g)$. Since both numbers g_1 and g_2 are even, this implies that (g_1, g_2) is a feasible pair.

Consider next the case where s is odd. If $g' \geqslant a(s) + 1$, then $g_1 = g' - 1 \geqslant a(s)$ and $g_2 = g' + 1 \geqslant a(s)$. Consequently, $H(g_1) + H(g_2) \geqslant 2\,\frac{7+s}{2} = 7 + s \geqslant F(g)$. Since both numbers g_1 and g_2 are even, this implies that (g_1, g_2) is a feasible pair.

Otherwise, $g' = a(s)$ and we argue as follows: since $g' \geqslant 3$, we have $s \geqslant 8$ and, by Proposition 4.2(f), $g' - 1 \geqslant a(s-2)$. Moreover, $g' + 1 \geqslant a(s)$. Consequently, $H(g' - 1) + H(g' + 1) \geqslant \frac{5+s}{2} + \frac{7+s}{2} = 6 + s$. If $F(g) = 6 + s$, then $(g' - 1, g' + 1)$ is a feasible pair. If $F(g) = 7 + s$, then $s = \left\lfloor \sqrt{12g} \right\rfloor$. Since $g' = a(s)$, this implies that $g' = a(s) = \left\lceil \frac{s^2}{24} \right\rceil$. Hence, by Proposition 4.2(e), there is an odd integer d' such that $g_1 = g' - d' = a(s) - d' \geqslant a(s-2)$ and $g_2 = g' + d' = a(s) + d' \geqslant a(s+2)$. Consequently, $H(g_1) + H(g_2) \geqslant \frac{5+s}{2} + \frac{9+s}{2} = 7 + s = F(g)$. Since both numbers g_1 and g_2 are even, this implies that (g_1, g_2) is a feasible pair. $\square$

5. Concluding remarks

Consider a surface Σ of Euler genus $g \geqslant 0$ and let G be a graph embedded on Σ. First suppose that G is the edge-disjoint union of two χ-critical graphs G_1 and G_2. Then Theorem 3.1 says that $\chi(G_1) + \chi(G_2) \leqslant H_2(\Sigma)$. For $g \geqslant 12$, the proof of this statement is only based on Euler's formula, Dirac's bound for the number of edges in χ-critical graphs (see Theorem 1.4) and Dirac's map colour theorem (see Theorem 2.5). The last two results also hold for the list-chromatic number χ^ℓ (see Theorems 1.4 and 2.6). Consequently, $\chi^\ell(2; \Sigma) = H_2(\Sigma)$ provided that $g = g(\Sigma) \geqslant 12$.

Now suppose that G is the edge-disjoint union of two χ-critical graphs G_1 and G_2 such that $\chi(G_1) + \chi(G_2) = H_2(\Sigma)$. Then it seems natural that both G_1 and G_2 are complete graphs. In the case where Σ is an orientable surface of Euler genus $g = g(\Sigma) \geqslant 20$, we can modify the proof of Theorem 3.1 slightly, to show that G_1 or G_2 is a complete graph. The question of whether both graphs are complete led us to the following conjecture.

Conjecture 5.1. *Let G be the edge-disjoint union of a complete graph K and an arbitrary graph H. Let H' be the graph obtained from H by contracting the set $V(K)$ to a single vertex. Then $g(H') + g(K) \leqslant g(G)$.*

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